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EXPLICIT EXPRESSIONS FOR CATALAN-DAEHEE NUMBERS

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Abstract. In this paper, we give some new and interesting identities and properties for the Catalan-Daehee numbers arising from p-adic invariant integrals on $\mathbb{Z}_p$.

1. Introduction

We note that the expression of $\sqrt{1+t}$ is given by

$$\sqrt{1+t}=\sum_{m=0}^{\infty}(-1)^{m-1}\binom{2m}{m}\left(\frac{1}{4^m}\right)\left(\frac{1}{2m-1}\right)t^m. \tag{1.1}$$

Replacing t by $-4t$ in (1.1), we have

$$\begin{aligned}\sqrt{1-4t}&=1-2\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{m+1}t^{m+1}\\&=1-2\sum_{m=0}^{\infty}C_mt^{m+1},\quad (\text{see [8]}),\end{aligned} \tag{1.2}$$

where C_m is the Catalan number.

The Catalan-Daehee numbers are defined by the generating function

$$\sum_{n=0}^{\infty}d_nt^n=\frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1},\quad (\text{see [9]}). \tag{1.3}$$

We observe that

$$\begin{aligned}&\frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1}\\&=\frac{1}{2}\frac{\log(1-4t)}{-4t}\left(\sqrt{1-4t}+1\right)\\&=\frac{1}{2}\sum_{l=0}^{\infty}\frac{4^l}{l+1}t^l\left(2-2\sum_{m=0}^{\infty}C_mt^{m+1}\right)\\&=\sum_{l=0}^{\infty}\frac{4^l}{l+1}t^l\left(1-\sum_{m=0}^{\infty}C_mt^{m+1}\right)\\&=\sum_{n=0}^{\infty}\frac{4^n}{n+1}t^n-\sum_{l=0}^{\infty}\frac{4^l}{l+1}t^l\sum_{m=0}^{\infty}C_mt^{m+1}\end{aligned} \tag{1.4}$$

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$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{4^n}{n+1} t^n - \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \frac{4^{n-m-1}}{n-m} C_m \right) t^n \\
&= 1 + \sum_{n=1}^{\infty} \left(\frac{4^n}{n+1} - \sum_{m=0}^{n-1} \frac{4^{n-m-1}}{n-m} C_m \right) t^n.
\end{aligned}
$$

Thus, by (1.3) and (1.4), we get

$$
\begin{aligned}
d_n &= \begin{cases} 1, & \text{if } n = 0, \\ \frac{4^n}{n+1} - \sum_{m=0}^{n-1} \frac{4^{n-m-1}}{n-m} C_m, & \text{if } n \geq 1 \end{cases} \\
&= - \sum_{m=0}^{n} \frac{4^{n-m}}{n-m+1} C_{m-1}^{*}, \quad \text{for all } n \geq 0,
\end{aligned} \tag{1.5}
$$

where

$$
C_{m-1}^{*} = \begin{cases} -1 & \text{if } m = 0, \\ C_{m-1} & \text{if } m \geq 1. \end{cases}
$$

We also define Catalan-Daehee polynomials $d_n(x)$ which are given by the generating function

$$
\frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1} (1-4t)^{\frac{x}{2}} = \sum_{n=0}^{\infty} d_n(x) t^n. \tag{1.6}
$$

Note that $d_n(0) = d_n$, $(n \geq 0)$.

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic ratioanl numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let $f(x)$ be a continuous $\mathbb{C}_p$-valued function on $\mathbb{Z}_p$. Then the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as

$$
\begin{aligned}
I_{-1}(f) &= \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) \\
&= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see [3–21]}).
\end{aligned} \tag{1.7}
$$

From (1.7), we note that

$$
I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \quad \text{where } f_1(x) = f(x+1). \tag{1.8}
$$

Thus, by (1.8), we get

$$
\begin{aligned}
\int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}}\, d\mu_{-1}(x) &= \frac{2}{1+\sqrt{1-4t}} \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} t^n \\
&= \sum_{n=0}^{\infty} C_n t^n, \quad (\text{see [8–17]}).
\end{aligned} \tag{1.9}
$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$.

On the other hand, for any uniformly differentiable function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic invariant integral $I_1(f)$ is given by

$$I_1(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [12, 13])}. \tag{1.10}$$

From (1.10), we can derive the following integral equation:

$$I_1(f_1) = I_1(f) + f'(0), \tag{1.11}$$

where $f_1(x) = f(x+1)$ and $f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}$.

Now, motivated by the relation in (1.9), Kim was led to consider

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}}\, d\mu(x) = \frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1} = \sum_{n=0}^{\infty} d_n t^n, \tag{1.12}$$

for $|t|_p < p^{-\frac{1}{p-1}}$ (see [9]).

Also,

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x+y}{2}}\, d\mu(y) = \frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1} (1-4t)^{\frac{1}{2}x} = \sum_{n=0}^{\infty} d_n(x)\, t^n. \tag{1.13}$$

In addition, it is amusing that $n!d_n(x)$ is the Sheffer sequence for $\left(g(x) = \frac{e^t-1}{t}, f(t) = \frac{1}{4}\left(1-e^{2t}\right)\right)$. That is, $n!d_n(x) \sim \left(\frac{e^t-1}{t}, \frac{1}{4}\left(1-e^{2t}\right)\right)$.

Recall that the Daehee numbers are given by the generating function

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad \text{(see [6])}. \tag{1.14}$$

In this paper, we give some new explicit expressions for Catalan-Daehee numbers which are derived from p-adic invariant integrals on $\mathbb{Z}_p$. In addition, we investigate some properties and identities for the Catalan-Daehee numbers arising from our explicit expressions.

2. Explicit expression for Catalan-Daehee numbers

From (1.6) and (1.14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} d_n t^n &= \left(\frac{\log(1-4t)}{-4t}\right)\frac{1}{2}\left(\sqrt{1-4t}+1\right) \\ &= \left(\sum_{l=0}^{\infty} (-4)^l D_l \frac{t^l}{l!}\right)\left(1 - \sum_{m=0}^{\infty} C_m t^{m+1}\right)\end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} (-4)^n D_n \frac{t^n}{n!} - \left(\sum_{l=0}^{\infty} (-4)^l D_l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} C_m t^{m+1} \right) \\
&= \sum_{n=0}^{\infty} (-4)^n D_n \frac{t^n}{n!} - \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1} C_m \right) t^n \\
&= 1 + \sum_{n=1}^{\infty} \left(\frac{(-4)^n}{n!} D_n - \sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1} C_m \right) t^n.
\end{aligned}
$$

Thus, by (2.1), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$
\begin{aligned}
d_n &= \begin{cases} 1, & \text{if } n = 0, \\ \frac{(-4)^n}{n!} D_n - \sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1} C_m, & \text{if } n \geq 1 \end{cases} \\
&= - \sum_{m=0}^{n} \frac{(-4)^{n-m}}{(n-m)!} D_{n-m} C_{m-1}^{*}.
\end{aligned} \tag{2.2}
$$

For $|t|_p < p^{-\frac{1}{p-1}}$, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} d_n t^n &= \int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}} \, d\mu(x) \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2} \right)^m \frac{1}{m!} \left(\log(1-4t) \right)^m \int_{\mathbb{Z}_p} x^m d\mu(x) \\
&= \sum_{m=0}^{\infty} 2^{-m} B_m \sum_{n=m}^{\infty} S_1(n,m) \frac{1}{n!} (-4t)^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} 2^{-m} B_m S_1(n,m) \frac{(-4)^n}{n!} \right) t^n,
\end{aligned} \tag{2.3}
$$

where B_n are the ordinary Bernoulli numbers defined by the generating function

$$
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.
$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$
d_n = \frac{(-1)^n}{n!} \sum_{m=0}^{n} 2^{2n-m} B_m S_1(n,m),
$$

where $S_1(n,m)$ is the Stirling number of the first kind.

Note that

$$
\begin{aligned}
\int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}} \, d\mu(x) &= \sum_{n=0}^{\infty} (-4)^n \int_{\mathbb{Z}_p} \binom{\frac{x}{2}}{n} d\mu(x) \, t^n \\
&= \sum_{n=0}^{\infty} d_n t^n.
\end{aligned} \tag{2.4}
$$

Thus, by (2.4), we obtain the following corollary.

Corollary 3. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \binom{\frac{x}{2}}{n} d\mu(x) = (-1)^n \frac{d_n}{4^n}, \tag{2.5}$$

and

$$\int_{\mathbb{Z}_p} \binom{\frac{x}{2}}{n} d\mu(x) = \frac{1}{n!} \sum_{m=0}^{n} 2^{-m} B_m S_1(n,m). \tag{2.6}$$

For $\lambda \in \mathbb{Z}_p$, $|t|_p < p^{-\frac{1}{p-1}}$, λ-Daehee polynomials $D_{n,\lambda}(x)$ are defined by the generating function

$$\frac{\lambda \log(1+t)}{(1+t)^\lambda - 1}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [7])}. \tag{2.7}$$

Thus, by (2.7), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+t)^{\lambda y + x} d\mu(y) &= \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1}(1+t)^x \\ &= \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.8}$$

In particular, λ-Daehee numbers $D_{n,\lambda} = D_{n,\lambda}(0)$ are given by

$$\begin{aligned} \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} &= \sum_{n=0}^{\infty} D_{n,\lambda} \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} (1+t)^{\lambda y} d\mu(y). \end{aligned} \tag{2.9}$$

From (2.8), we have

$$\begin{aligned} \sum_{n=0}^{\infty} d_n t^n &= \frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1} \\ &= \sum_{n=0}^{\infty} D_{n,\frac{1}{2}} (-4)^n \frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

Thus, by (2.10), we get

$$d_n = \frac{(-4)^n}{n!} D_{n,\frac{1}{2}}, \quad (n \geq 0). \tag{2.11}$$

Replacing t by $\frac{1}{4}\left(1-e^{2t}\right)$ in (1.12), we have

$$\begin{aligned} \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} &= \sum_{n=0}^{\infty} \frac{(-1)^n d_n}{4^n} \left(e^{2t}-1\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n d_n n!}{4^n} \sum_{m=n}^{\infty} S_2(m,n) \frac{2^m t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} (-1)^n 2^{m-2n} n! S_2(m,n) d_n \right) \frac{t^m}{m!}. \end{aligned} \tag{2.12}$$

Therefore, we obtain the following theorem.

Theorem 4. *For $m \geq 0$, we have*

$$B_m = \sum_{n=0}^{m} (-1)^n 2^{m-2n} n! S_2(m,n) d_n, \tag{2.13}$$

where $S_2(m,n)$ is the Stirling number of the second kind.

It is not difficult to show that

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x+y}{2}} d\mu(y) = \sum_{n=0}^{\infty} d_n(x) t^n = \sum_{n=0}^{\infty} D_{n,\frac{1}{2}}\left(\frac{x}{2}\right)(-4)^n \frac{t^n}{n!}. \tag{2.14}$$

Thus, by (2.14), we get

$$\int_{\mathbb{Z}_p} \binom{\frac{x+y}{2}}{n} d\mu(y) = \frac{d_n(x)}{(-4)^n}, \quad d_n(x) = \frac{(-4)^n}{n!} D_{n,\frac{1}{2}}\left(\frac{x}{2}\right), \quad (n \geq 0). \tag{2.15}$$

We observe that

$$\begin{aligned}
&(1-4t)^{\frac{x}{2}} \\
&= \sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^l \frac{1}{l!} (\log(1-4t))^l \\
&= \sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^l \sum_{m=l}^{\infty} S_1(m,l)(-4)^m \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} S_1(m,l) \frac{(-4)^m}{m!} \left(\frac{x}{2}\right)^l \right) t^m
\end{aligned} \tag{2.16}$$

Thus, by (2.16), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} d_n(x) t^n \\
&= \frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1} (1-4t)^{\frac{x}{2}} \\
&= \left(\sum_{k=0}^{\infty} d_k t^k \right) \left(\sum_{m=0}^{\infty} \sum_{l=0}^{m} S_1(m,l) \frac{(-4)^m}{m!} \left(\frac{x}{2}\right)^l t^m \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{l=0}^{m} S_1(m,l) \frac{(-4)^m}{m!} d_{n-m} \left(\frac{x}{2}\right)^l \right) t^n.
\end{aligned} \tag{2.17}$$

Therefore, by (2.17), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$\begin{aligned}
d_n(x) &= \sum_{m=0}^{n} \sum_{l=0}^{m} (-1)^m \frac{2^{2m-l}}{m!} S_1(m,l) d_{n-m} x^l \\
&= \sum_{l=0}^{n} \left(\sum_{m=l}^{n} (-1)^m \frac{2^{2m-l}}{m!} S_1(m,l) d_{n-m} \right) x^l.
\end{aligned} \tag{2.18}$$

We note that

$$\begin{aligned}(2.19)\qquad &\int_{\mathbb{Z}_p}(1-4t)^{\frac{y+x}{2}}\,d\mu(y)\\ &=\sum_{m=0}^{\infty}2^{-m}\frac{1}{m!}\left(\log(1-4t)\right)^m\int_{\mathbb{Z}_p}(y+x)^m\,d\mu(y)\\ &=\sum_{m=0}^{\infty}2^{-m}\frac{1}{m!}\left(\log(1-4t)\right)^m B_m(x)\\ &=\sum_{m=0}^{\infty}2^{-m}B_m(x)\sum_{n=m}^{\infty}S_1(n,m)\frac{(-4)^n}{n!}t^n\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}2^{-m}B_m(x)\,S_1(n,m)\frac{(-4)^n}{n!}\right)t^n,\end{aligned}$$

where $B_m(x)$ are the Bernoulli polynomials.

Therefore, by (1.13) and (2.19), we obtain the following theorem.

Theorem 6. *For $n\geq 0$, we have*

$$d_n(x)=\frac{(-4)^n}{n!}\sum_{m=0}^{n}2^{-m}S_1(n,m)\,B_m(x).$$

From (1.6), we have

$$\begin{aligned}(2.20)\qquad &\sum_{n=0}^{\infty}d_n(x)\,t^n\\ &=\frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1}(1-4t)^{\frac{x}{2}}\\ &=\frac{\log(1-4t)}{-4t}(1-4t)^{\frac{x}{2}}\frac{1}{2}\left(\sqrt{1-4t}+1\right)\\ &=\left(\sum_{l=0}^{\infty}(-4)^l D_l\left(\frac{x}{2}\right)\frac{t^l}{l!}\right)\left(1-\sum_{m=0}^{\infty}C_m t^{m+1}\right)\\ &=\sum_{l=0}^{\infty}(-4)^l D_l\left(\frac{x}{2}\right)\frac{t^l}{l!}-\left(\sum_{l=0}^{\infty}(-4)^l D_l\left(\frac{x}{2}\right)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}C_m t^{m+1}\right)\\ &=\sum_{n=0}^{\infty}(-4)^n D_n\left(\frac{x}{2}\right)\frac{t^n}{n!}-\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}(-4)^{n-m-1}D_{n-m-1}\left(\frac{x}{2}\right)\frac{C_m}{(n-m-1)!}t^m\\ &=D_0\left(\frac{1}{2}x\right)-\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\frac{(-4)^{n-m-1}D_{n-m-1}\left(\frac{x}{2}\right)}{(n-m-1)!}C_m^*\right)t^n\\ &=-\sum_{n=0}^{\infty}\left(\sum_{m=-1}^{n-1}\frac{(-4)^{n-m-1}D_{n-m-1}\left(\frac{x}{2}\right)}{(n-m-1)!}C_m^*\right)t^n\\ &=-\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{(-4)^{n-m}}{(n-m)!}D_{n-m}\left(\frac{x}{2}\right)C_{m-1}^*\right)t^n.\end{aligned}$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 7. *For $n \geq 0$, we have*

$$d_n(x) = -\sum_{m=0}^{n} \frac{(-4)^{n-m}}{(n-m)!} C_{m-1}^{*} D_{n-m}\left(\frac{x}{2}\right).$$

Remark. We note that

$$\begin{aligned} \sum_{n=0}^{\infty} d_n(x) t^n &= \frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1}(1-4t)^{\frac{1}{2}x} \\ &= \left(\sum_{l=0}^{\infty} d_l t^l\right)\left(\sum_{m=0}^{\infty} \binom{\frac{1}{2}x}{m}(-4)^m t^m\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \binom{\frac{1}{2}x}{m}(-4)^m d_{n-m}\right) t^n. \end{aligned} \tag{2.21}$$

Thus, by (2.21), we get

$$d_n(x) = \sum_{m=0}^{n} \binom{\frac{1}{2}x}{m}(-4)^m d_{n-m}, \quad (n \geq 0).$$

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